Int. J. Solids Structures, 1975, Vol. 11, pp. 871–876. Pergamon Press. Printed in Great Britain.

THE INFLUENCE OF ELECTRODES ON THE RESONANCE FREQUENCY OF AT-CUT QUARTZ PLATES

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(*Received* 4 *September* 1974; *revised* 26 *December* 1974)

Abstract—The shear-flexural-twist vibrations in rectangular AT-cut quartz plates with partial electrodes are considered and the influences of metalic electrodes on the resonance frequency and on the resonance frequency temperature dependence are calculated. Different elastic stiffnesses were supposed in the plated and unplated part of the plate and in this way the influence of piezoelectricity is considered. The influences of mass loading and the length of electrodes have been calculated and examples of computed results are given.

1. INTRODUCTION

The piezoelectric AT-cut quartz resonators are the most frequent type of piezoelectric resonators today. They are used in the frequency range from 0·8 to 200 MHz and work on the principle of the thickness-shear mode of vibration. Increasing requirements concerning especially the temperature dependence of the resonance frequency of these resonators make it necessary to know in detail the influence of various parameters on the resonance frequency. First of all it is necessary to determine with sufficient accuracy the influence of the shape and dimensions of the resonator and the layout, shape and dimensions of electrodes. In the paper the possibility of theoretical solving of this problems is indicated for the case of perpendicular resonators with perpendicular electrodes.

The two-dimensional approximate method indicated by Mindlin[1] are taken into account when the relations describing vibrations of AT-cut quartz plates are derived. There is also suppose in the paper that the thickness-shear vibrations of boundered AT-cut quartz plates are influenced by elastic coupling between thickness-shear and flexural vibrations[2]. The equations of motion of coupled flexural and thickness-shear vibrations AT-cut resonators are therefore considered[3].

Mindlin and Gazis[2] and Mindlin and Spencer[3] considered otherwise boundered but not plated quartz AT-cut resonators. Relations which they derived are suitable for accurate resonance frequency calculation only in the case that the vibration are excited by an alternating electric field which is produced between electrodes perpendicular to the x_2 axis and separated from the plate by a sufficiently large air gap.

Mindlin[4] derived the equations of motion and the frequency equation where the mass loading given by electrodes deposition was considered. Byrne, Lloyd and Spencer[5] and Lee and Spencer[6] went from Mindlin's paper out and studied the vibration of partially AT-cut quartz plates. They took into account the influence of electrodes the same way as Mindlin. In comparison with Lee and Spencer's paper[6] in this paper the piezoelectric stiffening and the temperature dependence of the resonance frequency of the plates with partial electrodes are also considered.

2. DESCRIPTION OF VIBRATIONS AND THE FREQUENCY EQUATION OF AT-CUT RESONATORS WITH PARTIAL ELECTRODES

The influence of infinitely thin and perfectly conducting electrodes on the resonance frequency of the resonator vibrated in thickness shear mode was studied by Lawson[7]. He pointed that with regard to piezoelectric properties of the plate the deformation on the surface of the plate is non zero and that the value of the deformation is a function of the air gap between the electrodes and the surface of the plate. Lawson took into account the influence of the air gap in the frequency equation by introducing effective thickness of the plate. The effective thickness is generally large than the real thickness of the plate and is equal to the real thickness only if the air gap is infinitely large.

The extreme cases of the distance of electrodes from the surface of the plate may be respected in the equations of motion and in the frequency equation also by substituting suitable elastic stiffnesses.

As may be seen from paper [7] the zero deformation is on the surface of the plate in the case of an infinitely large air gap. This situation corresponds to the conditions considered when solving piezoelectrically stiffened vibrations, where in the case of thickness-shear vibration the value

$$c_{66} = c^E_{66} + \frac{e^2_{26}}{\epsilon_{22}} = c^D_{66} \tag{1}$$

is substituted for the elastic stiffness c_{66} in the equations of motions[8, 9]. In equation (1) c^E_{66} is the elastic stiffness measured at a constant electric field intensity E, c^D_{66} is the elastic stiffness measured at a constant electric displacement D, e_{26} is piezoelectric modulus corresponding to the electric field in x_2 axis direction and thickness-shear deformation in the plane x_1x_2 and ϵ_{22} is permitivity in the direction of x_2 axis.

In case of a non zero air gap and short-circuited electrodes (this may be approximately fulfilled also by electrodes shunted by small resistance) the non zero deformation give rise to an electric charge on the surface of the plate which compensates the corresponding components of the electric field so that in its plated part the plate behaves as piezoelectricaly unstiffened and elastic stiffness c_{66} has the value c^E_{66}.

If we consider these two extreme cases then we may take into account piezoelectric properties of partially plated resonators so that in the unplated part of the plate we shall express the elastic quality by elastic stiffness c^D_{66} and in the plated part of the plate by elastic stiffness c^E_{66}. Let us suppose that the AT-cut quartz plate is plated and orientated in an orthogonal system of axes in a way shown in Fig. 1. Then we may write the equations of motion if the piezoelectric stiffering and mass loading are considered in similar form as in the papers [3] and [6] but with diperent elastic stiffnesses in unplated and plated part of the plate.

Unplated part

$$\begin{gathered} K_1^2 c^D_{66}(u_{2,11} + \psi_{1,1}) - \tfrac{1}{3}h^2\gamma_{55}(u_{2,1133} - \psi_{1,133}) = \rho\ddot{u}_2 \\ \gamma_{11}\psi_{1,11} - \gamma_{55}(u_{2,133} - \psi_{1,33}) - 3h^{-2}K_1^2 c^D_{66}(u_{2,1} + \psi_1) = \rho\ddot{\psi}_1 \end{gathered} \tag{2}$$

Plated part (the quantities are marked with prime)

$$\begin{gathered} \bar{K}_1^2 c^E_{66}(\bar{u}_{2,11} + \bar{\psi}_{1,1}) - \tfrac{1}{3}h^2\gamma_{55}(\bar{u}_{2,1133} - \bar{\psi}_{1,133}) = \rho(1+R)\ddot{\bar{u}}_2 \\ \gamma_{11}\bar{\psi}_{1,11} - \gamma_{55}(\bar{u}_{2,133} - \bar{\psi}_{1,33}) - 3h^{-2}\bar{K}_1^2 c^E_{66}(\bar{u}_{2,1} + \bar{\psi}_1) = \rho(1+3R)\ddot{\bar{\psi}}_1 \end{gathered} \tag{3}$$

where R is mass loading of the plated part of the plate defined as the ratio of the mass per unit area of both electrodes to the mass per unit area of the plate

$$R = \frac{\rho' h'}{\rho h} \tag{4}$$

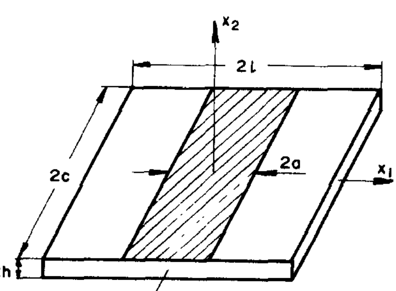

Fig. 1. The position of the rectangular AT-cut quartz plate with an electrode strip in the orthogonal coordinate system.

ρ' is the density of electrodes and h' is the total thickness of both electrodes.

The correction factors K_1 and $\bar{K}_1$ in equations (2) and (3) are given by relations

$$K_1^2 = \frac{\pi^2}{12}, \quad \bar{K}_1^2 = \frac{\pi^2}{12}\frac{1+3R}{(1+R)^2}. \tag{5}$$

When a uniform electric field is impressed over the electrode platings and when at the boundaries of the plated and unplated portions of the plate the conditions given in [6] are fulfilled then for the plated part of the plate we may similarly as in [6] suppose the solution of equations (3) in the form

$$\begin{aligned} \bar{u}_2 &= (\bar{A}_1 h \sin \bar{\xi}_1 x_1 + \bar{A}_2 h \sin \bar{\xi}_2 x_1) \cos \bar{\zeta} x_3 \, e^{j\omega t} \\ \bar{\psi}_1 &= (\bar{B}_1 \cos \bar{\xi}_1 x_1 + \bar{B}_2 \cos \bar{\xi}_2 x_1) \cos \bar{\zeta} x_3 \, e^{j\omega t} \end{aligned} \tag{6}$$

in unplated part of the plate we may consider the solution of equations (2) in the form

$$\begin{aligned} u_2 = [&A_{11} h \sin \xi_1(x_1 - a) + A_{12} h \cos \xi_1(x_1 - a) \\ &+ A_{21} h \sin \xi_2(x_1 - a) + A_{22} h \cos \xi_2(x_1 - a)] \cos \zeta x_3 \, e^{j\omega t} \\ \psi_1 = [&B_{11} \cos \xi_1(x_1 - a) + B_{12} \sin \xi_1(x_1 - a) + B_{21} \cos \xi_2(x_1 - a) + B_{22} \sin \xi_2(x_1 - a)] \cos \zeta x_3 \, e^{j\omega t} \end{aligned} \tag{7}$$

where $u_2(x_1, x_3)$ is the deflection of the plate element and $\psi_1(x_1, x_3)$ is the rotation of a line element about x_3 axis.

From the requirement of the continuity of the displacements and stresses at the boundaries of the plated and unplated part of the plate ($x_1 = \pm a$) it follows that

$$u_2 = \bar{u}_2, \quad \psi_1 = \bar{\psi}_1, \quad \psi_{1,1} = \bar{\psi}_{1,1}.$$

Then the frequency equation may be written in the same form as in Lee and Spencer's paper [6]

$$|a| = \begin{vmatrix} a_{11} & a_{12} & 0 & \phi_1 & 0 & \phi_2 \\ a_{21} & a_{22} & 1 & 0 & 1 & 0 \\ a_{31} & a_{32} & \sigma_1 & 0 & \sigma_2 & 0 \\ a_{41} & a_{42} & 0 & -\alpha_1 & 0 & -\alpha_2 \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix} = 0 \tag{8}$$

Also the relations for the calculation of the components of the frequency equation (8) are given in paper [6] but for the calculation of the quantities given below it is necessary to use the following relations

$$\bar{\phi}_i = \bar{\xi}_i h \qquad \phi_i = \xi_i h$$

$$\bar{x} = \bar{\zeta} h \qquad x = \zeta h$$

$$\bar{\omega}_1^2 = \frac{3\bar{K}_1 c_{66}^E}{\rho(1+3R)h^2} \qquad \omega_1^2 = \frac{3K_1 c_{66}^D}{\rho h^2}$$

$$\bar{\gamma}_{55} = \frac{\gamma_{55}}{3\bar{K}_1^2 c_{66}^E} \qquad \hat{\gamma}_{55} = \frac{\gamma_{55}}{3K_1^2 c_{66}^D}$$

$$a^* = \frac{a}{h} \qquad d^* = \frac{l-a}{h}$$

$$\bar{\gamma}_{11} = \frac{\gamma_{11}}{3\bar{K}_1^2 c_{66}^E} \qquad \hat{\gamma}_{11} = \frac{\gamma_{11}}{3K_1^2 c_{66}^D}$$

where the symbols used here are the same as in Lee and Spencer's paper [6].

Table 1. Values and temperature coefficients of the elastic stiffnesses used in the calculation of resonance frequencies and the resonance frequency temperature dependence (taken from [11–13])

$\lambda\mu$	Values for D = constant		Values for E = constant		The same values for E = constant, D = constant	
	$c^D_{\lambda\mu}$ 10^9 N m^{-2}	$Tc^{(1)}_{\lambda\mu}$ 10^{-6} /°C	$c^E_{\lambda\mu}$ 10^9 N m^{-2}	$Tc^{(1)}_{\lambda\mu}$ 10^{-6} /°C	$Tc^{(2)}_{\lambda\mu}$ 10^{-9} /°C^2	$Tc^{(3)}_{\lambda\mu}$ 10^{-12} /°C^3
11	87.49	-46.8	86.74	-44.3	-107	-70
12	6.23	-2975.0	6.99	-2690.0	-3050	-1260
13	11.94	-550.0	11.91	-550.0	-1150	-750
14	-18.09	100.0	-17.91	117.0	-48	-590
33	107.2	-160.0	107.2	-160.0	-275	-250
44	57.98	-177.4	57.94	-175.4	-216	-216
66	40.63	177.7	39.88	187.6	118	21

3. THE TEMPERATURE DEPEDENCE OF THE RESONACE FREQUENCY OF PARTIALLY PLATED AT-CUT QUARTZ PLATES

The possibility to use the frequency equation of coupled flexural and thickness shear vibrations derived by using two dimensional approximation method was given in paper [10] when unplated AT-cut quartz plates were considered. The same way has been also used in the calculation of the resonance frequency temperature dependence of partially plated AT-cut quartz resonators. The basic physical constants and dimensions of the plate (this is c^D_{66}, c^E_{66}, γ_{11}, γ_{55}, ρ, $2h$, $2l$) were calculated first of all for a chosen number of temperatures T_i according to the relations

$$[y]_{T_i} = [y]_{T_0}\left[1 + \sum_{n=1}^{3} T_y^{(n)}(T_i - T_0)^n\right] \tag{9}$$

where $[y]_{T_i}$ is the value of the quantity at the temperature T_i, $[y]_{T_0}$ is the value of the quantity at the temperature T_0 and $Ty^{(1)}$, $Ty^{(2)}$, $Ty^{(3)}$ are the first, second and third order frequency temperature coefficients.

The values of the elastic stiffnesses temperature coefficients used in the calculation were taken from papers [11], [12] and [13] and are shown here in Table 1.

The temperature dependence of the resonance frequency has been presented by means of the temperature dependence of the frequency constant k_f given by the relation

$$\bar{K}_f = \frac{\bar{\Omega}}{\pi}\sqrt{\left(\frac{3\bar{K}_1^{\,2}}{\rho(1+3R)}c^E_{66}\right)}. \tag{10}$$

As the resonance frequency is given as the ratio of the frequency constant $\bar{K}_f$ and the thickness of the plate h, the frequency constant is divided by the coefficient expressing the change of the plate thickness depending on temperature

$$[\bar{K}_f]_{T_i} = \bar{K}_f \frac{1}{1 + \sum_{n=1}^{3} Th^{(n)}(T_i - T_0)^n}. \tag{11}$$

4. RESULTS OF CALCULATION

The AT-cut quartz square plate with orientation yxl 35°18′ was considered in the calculation. The length over thickness ratio of the plate was chosen 20. The influence of the mass loading R and the length of the electrodes $2a$ was considered in the ranges $R = 0$–$0{\cdot}04$, $a/h = 2{\cdot}5, 5, 7{\cdot}5, 10$.

First of all the dependence of the frequency constant on the mass loading R was studied. The calculated dependence for $a/h = 5$ and 10 is given in Fig. 2.

For the comparison of the theoretical and measured results is better to use instead of the mass

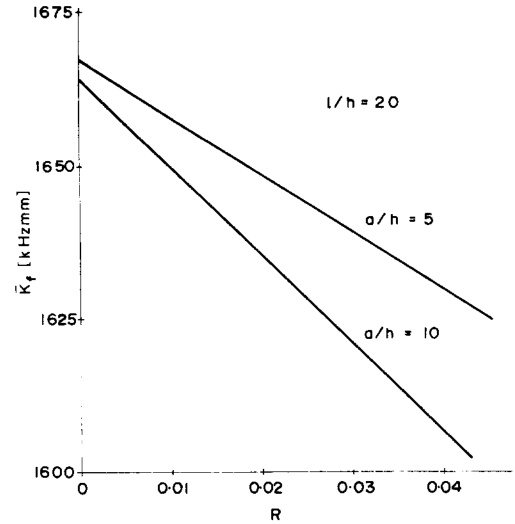

Fig. 2. Calculated dependence of the frequency constant K_f of the rectangular AT-cut quartz plate with orientation *yxl* 35°18′ on the mass loading *R*.

loading R the plate back Δ. The plate back Δ is given by relation

$$\Delta = \frac{\bar{K}_{fn} - \bar{K}_f}{\bar{K}_f} \tag{12}$$

where $\bar{K}_{fn}$ is the frequency constant of the unplated plate. For the same value of R the plate back Δ is function of the length of the electrodes. The influence of the electrode length upon the plate back Δ is seen from the Table 2 where the calculated and measured values of the plate back of AT-cut quartz square plates with orientation *yx* 35°17′ and nominal resonance frequency 10·3 MHz is given for two length of the electrodes. From Table 2 is also seen good agreement between measured and calculated results.

The influence of the electrodes on the resonance frequency of piezoelectric bars and AT-cut quartz plates was experimentally studied by Suk[14]. He considered only circular AT-cut quartz plates. But it is possible to state that the comparison of the character of the theoretical results for AT-cut square plates and experimental results obtained for AT-cut circular plates is very good.

The calculated temperature dependence of the resonance frequency is given in Fig. 3 for $a/h = 5$ and 10 and $R = 0$ and 0·04. It was observed during calculation that the mass loading smaller than $R < 0{\cdot}01$ does not have visible influence on the temperature dependence. Only for $R \geqslant 0{\cdot}03$ the influence of the electrodes is expressive. The calculated temperature dependence of the resonance frequency has the same character as the temperature dependence obtained experimentally by Miller[15] and Spencer[17].

5. CONCLUSIONS

An attempt of a more accurate theoretical expression of the influence of electrodes on the resonance frequency partially plated AT-cut quartz resonators was shortly described in the

Table 2. Calculated frequency constants and calculated and measured plate back Δ of the AT-cut quartz square plates $l/h = 43{\cdot}75$ with orientation *yxl* 35°17′ and $R = 0{\cdot}0066$

The length of the electrodes a/l /mm/	$\bar{K}_f$ (calculated) /KHz mm/	Δ /%/	
		calculated	measured
unplated	1665.984	-	-
12.5	1650.800	0.920	0.95
18.75	1647.997	1.091	1.28

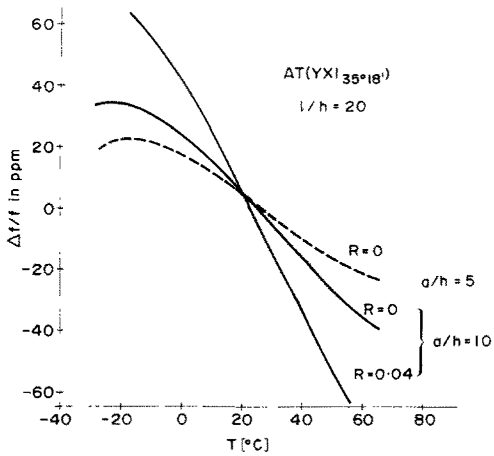

Fig. 3. Calculated resonance frequency temperature dependence of the rectangular AT-cut quartz plate for the different mass loading *R*.

paper. Also a possibility was indicated to use this more accurate expression for the calculation of the influence of electrodes on the temperature dependence of the resonance frequency. In both cases not only the mass loading but also the piezoelectric stiffering were taken into account. The preliminary comparison of the results calculated for the AT-cut square resonators with the results obtained experimentally for the AT-cut circular resonators gave a good qualitative agreement. It may be supposed that the described theoretical expression of the influence of electrodes on the resonance frequency of AT-cut quartz plates will make it possible for designing of resonators to respect more accurately the influence of electrodes.

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